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INTEGRATION OF THE EQUATIONS OF A ROTATIONAL MOTION OF A RIGID BODY IN QUATERNION ALGEBRA. THE EULER CASE†

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A dynamical system is constructed in the multiplicative group of the quarternion algebra **H** that serves as the configuration space. A homomorphism $\mathbf{H} \to SO(3)$ is used such that the unit sphere $S^3 \subset \mathbf{H}$, invariant under the system, is transformed into the rotation group $SO(3)$. The homomorphic image of the system is identical with the dynamics of rotational motion of a rigid body. The equations of motion are completely integrated in the Euler case. To this end Weierstrass' elliptic functions are used. The following goals are achieved within the framework of the method: (a) when representing the algorithms for modelling the dynamics it suffices to use only one chart from the atlas of the phase space manifold, (b) the point in the configuration space of the actual motion lies on the unit sphere, which ensures the best accuracy in numerical procedures, and (c) in the majority of applications the right-hand sides of the equations of perturbed motion depend polynomially on the phase variables, which simplifies the use of computer algebra in analytic theories.

1. INTRODUCTION

To construct the equations of perturbed motion in rigid body dynamics it is most convenient from the point of view of computations to use a symmetric representation of the configuration space in terms of the Rodrigues–Hamilton or Cayley–Klein parameters [14].

As the configuration space we shall use the quaternion algebra **H**[4], the four-dimensional Euclidean space $\mathbf{R}^4$ being the coordinate space. More precisely, the algebra **H** without the null point, that is, $M^4 = \mathbf{H}\backslash\{0\}$, is the configuration manifold. This manifold can obviously be covered by a single chart $\mathbf{R}^4\backslash\{0\}$. Clearly, M^4 is the multiplicative group of **H**. It is well known that the group $SO(3)$ of rotations in $\mathbf{R}^3$ is the configuration space of a rigid body with a fixed point. The smooth manifold $SO(3)$ is diffeomorphic to $\mathbf{RP}^3$ that inherits the smooth structure of the sphere S^3 with identified antipodal points. The manifold S^3 can be identified with the surface of the unit sphere in $\mathbf{R}^4$ defined by the equation

$$|\mathbf{q}|^2 = 1 \quad \left(|\mathbf{q}|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2\right) \tag{1.1}$$

The unit vectors of the coordinate axes in $\mathbf{R}^4$ will be denoted by $\mathbf{1}, \mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$, so that an arbitrary vector $\mathbf{q} \in \mathbf{R}^4$ can be represented as

$$\mathbf{q} = q_0\mathbf{1} + q_1\mathbf{i}_1 + q_2\mathbf{i}_2 + q_3\mathbf{i}_3$$

The quaternion algebra **H** is defined in $\mathbf{R}^4$ in such a way that $\mathrm{Re}\,\mathbf{q} = q_0\mathbf{1}$ is the scalar part of a quaternion and $\mathrm{Ve}\mathbf{q} = q_1\mathbf{i}_1 + q_2 i_2 + q_3\mathbf{i}_3$ is its vector part. The rotational motion of a rigid body can be described by a Lagrangian system on SO^3. This system admits of a well-defined extension to S^3, so that the kinetic energy and force field are also correctly defined.

In the majority of cases the moments of forces do not depend on the direction cosines or the angular velocities. We shall extend the force field from S^3 to M^4 in such a way that the components of the generalized forces at a point on a sphere S_R^3 of arbitrary radius R will be equal to the corresponding components at the point on the unit sphere $S^3 = S_1^3$ that belongs to the same ray issuing from the origin in $\mathbf{R}^4$.

2. EQUATIONS OF MOTION

In M^4 we introduce a local curvilinear coordinate system by the formulae

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$$q_0 = e^{\alpha_0}\cos\frac{\alpha_2}{2}\cos\frac{\alpha_1+\alpha_3}{2},\quad q_1 = e^{\alpha_0}\sin\frac{\alpha_2}{2}\cos\frac{\alpha_1-\alpha_3}{2}$$
$$q_2 = e^{\alpha_0}\sin\frac{\alpha_2}{2}\sin\frac{\alpha_1-\alpha_3}{2},\quad q_3 = e^{\alpha_0}\cos\frac{\alpha_2}{2}\sin\frac{\alpha_1+\alpha_3}{2} \tag{2.1}$$
$$\alpha_0\in(0,\infty),\quad \alpha_2\in[0,\pi),\quad \alpha_1,\alpha_3\in[0,2\pi)$$

Then the parameters α_1, α_2, α_3 on S^3 (and also on $SO(3) \simeq \mathbf{RP}^3$) can be interpreted as the Euler angles: precession, nutation and rotation, respectively. The orthogonal transition matrix U from the stationary system coordinates of $O\xi_1, \xi_2, \xi_3$ to the system Ox_1, x_2, x_3 attached to the body (the rigid body having a fixed point O) is given by

$$U = T_3(\alpha_1)T_1(\alpha_2)T_3(\alpha_3)$$

where the matrices

$$T_1(\alpha) = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{Vmatrix},\quad T_3(\alpha) = \begin{Vmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{Vmatrix}$$

correspond to rotation by an angle α about the first and third coordinate axes. The columns of U are formed by the coordinates of the basis vectors of the stationary system Ox_1, x_2, x_3 relative to the system $O\xi_1, \xi_2, \xi_3$. We assume that Ox_1, x_2, x_3 are the principal axes of inertia. The elements of U are the direction cosines such that

$$U(\mathbf{q}) = \frac{1}{|\mathbf{q}|^2}\begin{Vmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2-q_0q_3) & 2(q_1q_3+q_0q_2) \\ 2(q_0q_3+q_1q_2) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3-q_0q_1) \\ 2(q_1q_3-q_0q_2) & 2(q_0q_1+q_2q_3) & q_0^2-q_1^2-q_2^2+q_3^2 \end{Vmatrix} \tag{2.2}$$

Note that $U(\mathbf{q})$ depends only on the points $\mathbf{q}/|\mathbf{q}| \in S^3$.

We know [4] that the transformation $g : \mathbf{q} \mapsto U(\mathbf{q})$ defines a two-sheeted covering $g : S^3 \to SO(3)$, which can be extended to a group homomorphism $g : \mathbf{H} \to SO(3)$.

Indeed, by analogy with [4], we define a homomorphism $h : \mathbf{H} \to SU(2)$ from the multiplicative group of the quaternion algebra to the special unitary group as follows:

$$h : \mathbf{q} \mapsto \frac{1}{|\mathbf{q}|}\begin{Vmatrix} q_0+iq_1 & q_2+iq_3 \\ -q_2+iq_3 & q_0-iq_1 \end{Vmatrix}$$

The map h is a diffeomorphism on the unit sphere $S^3 \subset \mathbf{H}$. It is well known [5] that there is a homomorphism $\sigma : SU(2) \to SO(3)$, which ensures a two-sheeted covering of $SO(3)$ by the group $SU(2)$, the kernel of which consists of two elements: $\ker\sigma = \{E, -E\}$. As a result, we obtain a homomorphism such that the diagram

$$\begin{tikzcd} \mathbf{H} \arrow[r, "h"] \arrow[dr, "g"'] & SU(2) \arrow[d, "\sigma"] \\ & SO(3) \end{tikzcd}$$

commutes.

Following [6, 7], we introduce the quasivelocities ω_0, ω_1, ω_2, ω_3 by the formula

$$\boldsymbol{\omega} = \frac{2}{|\mathbf{q}|^2}S^*(\mathbf{q})\mathbf{q}^{\cdot} \tag{2.3}$$

$$\boldsymbol{\omega} = \begin{Vmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{Vmatrix},\quad S(\mathbf{q}) = \begin{Vmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{Vmatrix},\quad \mathbf{q}^{\cdot} = \begin{Vmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{Vmatrix}$$

(the asterisk denotes transposition). Then the kinetic energy of the mechanical system can be computed in the form [6]

$$T = \frac{1}{2}(A\boldsymbol{\omega}, \boldsymbol{\omega}); \quad \mathrm{A} = \mathrm{diag}(A_0, A_1, A_2, A_3), \quad A_0 = \frac{1}{2}(A_1 + A_2 + A_3) \tag{2.4}$$

where A is the matrix of the tensor of 'inertia', A_1, A_2, A_3 and the moments of inertia about the axes Ox_1, Ox_2, Ox_3; and where $(\cdot, \cdot)$ denotes the scalar product in $\mathbf{R}^4$.

If M_1, M_2, M_3 are the projections of the principal moment of forces acting on the body on to the axes Ox_1, Ox_2, Ox_3, then the Lagrange equations of the second kind are equivalent to the extended system of Euler dynamic equations

$$\mathbf{I}^{\cdot} + \frac{1}{2}(\boldsymbol{\omega} \circ \mathbf{I} - \mathbf{I} \circ \boldsymbol{\omega}) = \mathbf{M} \tag{2.5}$$

The vector $\mathbf{I} = A\boldsymbol{\omega}$ consists of the components of the moment-of-momentum vector, $\mathbf{M} = (0, M_1, M_2, M_3)$, and the centred degree symbol denotes group multiplication in the algebra $\mathbf{H}$. We need to supplement (2.5) by a system resembling the kinematic equations.

$$\mathbf{q}^{\cdot} = \frac{1}{2}\mathbf{q} \circ \boldsymbol{\omega} \tag{2.6}$$

The advantage of system (2.5), (2.6) is that it has the Cauchy form and the right-hand sides depend polynomially on q_k, ω_k $(k = 0, 1, 2, 3)$. In applications the moments M_k $(k = 0, 1, 2, 3)$ are often polynomial functions of the direction cosines. By (2.2) the right-hand sides of (2.5) contain terms with $|\mathbf{q}|^{-2}$ as a multiplier. However, it should be observed that real motion takes place on the configuration submanifold (1.1).

Manifolds of the form $|\mathbf{q}|^{-2}$ = const are invariant under the dynamical system (2.5), (2.6). In other words, (1.1) is an invariant relation. Clearly, if $|\mathbf{q}| = 1$ is substituted everywhere into the right-hand sides of (2.5), the resulting dynamical system will be identical with the original one on (1.1), i.e. it will describe the dynamics of a rigid body.

Therefore, the following result holds.

Proposition. If the moments of forces are polynomial functions of the direction cosines and the components of the angular velocity, then the dynamic and kinematic Euler equations admit of an extension involving the Rodrigues–Hamilton parameters, for which they attain the Cauchy form with polynomial right-hand sides.

The canonical form of the equations of motion may turn out to be useful in modelling problems. Using a Legendre transformation one can obtain an expression for the generalized moments [6]

$$\mathbf{p} = (p_0, p_1, p_2, p_3)^* = 2S(\mathbf{q})\mathbf{I}/|\mathbf{q}|^2$$

Then the quasivelocities can be expressed in terms of the canonical variables as

$$\mathbf{I} = \frac{1}{2}S^*(\mathbf{q})\mathbf{p} \tag{2.7}$$

Substituting (2.7) into (2.4), we obtain an expression for the kinetic energy

$$T = (8A_0)^{-1}(q_0p_0 + q_1p_1 + q_2p_2 + q_3p_3)^2 + (8A_1)^{-1}(-q_1p_0 + q_0p_1 + q_3p_2 - q_2p_3)^2 +$$
$$+(8A_2)^{-1}(-q_2p_0 - q_3p_1 + q_0p_2 + q_1p_3)^2 + (8A_3)^{-1}(-q_3p_0 + q_2p_1 - q_1p_2 + q_0p_3)^2$$

The generalized forces can also be represented as

$$\mathbf{Q} = (Q_0, Q_1, Q_2, Q_3)^* = 2S(\mathbf{q})\mathbf{M}/|\mathbf{q}|^2$$

As a result, the system of Hamilton equations takes the form

$$q_k^{\cdot} = \partial T / \partial p_k, \quad p_k^{\cdot} = -\partial T / \partial q_k + Q_k, \quad k = 0,1,2,3 \tag{2.8}$$

In what follows we shall assume that there is a force function $V(q)$ such that

$$Q_k = \partial V / \partial q_k, \ k = 0, 1, 2, 3$$

In this case the Hamilton function

$$H(\mathbf{q}, \mathbf{p}) = T(\mathbf{q}, \mathbf{p}) + V(\mathbf{q})$$

exists.

By (2.7) we can use the convector $\mathbf{I} \in T_{\mathbf{q}}^* M^4$ instead of the covector $\mathbf{p} \in T_{\mathbf{q}}^* M^4$ in the phase space $T^* M^4$ for a fixed $\mathbf{q} \in M^4$. We shall use two symbols for the Hamilton function

$$H(\mathbf{q}, \mathbf{p}) = F(\mathbf{q}, \mathbf{I})$$

By lengthy transformations one can obtain the system of different equations

$$\begin{Vmatrix} \mathbf{q}^{\cdot} \\ \mathbf{I}^{\cdot} \end{Vmatrix} = \begin{Vmatrix} 0 & \frac{1}{2} S(\mathbf{q}) \\ -\frac{1}{2} S^*(\mathbf{q}) & P(\mathbf{I}) \end{Vmatrix} \begin{Vmatrix} \partial F / \partial \mathbf{q} \\ \partial F / \partial \mathbf{I} \end{Vmatrix} \tag{2.9}$$

$$P(\mathbf{I}) = 2 \begin{Vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -I_3 & I_2 \\ 0 & I_3 & 0 & -I_1 \\ 0 & -I_2 & I_1 & 0 \end{Vmatrix}$$

Remark. Equations (2.8) and (2.9) also admit of the transformation described in the proposition. Namely, putting $| \mathbf{q} |$ equal to unity in the denominators in Q_k ($k = 0, 1, 2, 3$), we can obtain a polynomial dependence of the generalized forces on the coordinates and momenta without distorting the motion. We then obtain a dynamical system in M^4, which differs, in general, from (2.8). For example, if Q_k admit of a force function, the new generalized forces do not have to be potential ones. However, this is unimportant from the viewpoint of using semianalytic and projection methods [8] for the approximate construction of solutions.

3. IMBEDDING

We shall establish a duality between the dynamical system under consideration and the system on $SO(3)$, defining the dynamics of a rigid body in the standard way. It can be verified that quasivelocities (2.3) can be represented as follows with the aid of (2.1):

$$\boldsymbol{\omega} = B(\boldsymbol{\alpha}) \boldsymbol{\alpha}^{\cdot}$$

$$B(\boldsymbol{\alpha}) = \begin{Vmatrix} 2 & 0 & 0 & 0 \\ 0 & \sin\alpha_2 \sin\alpha_3 & \cos\alpha_3 & 0 \\ 0 & -\sin\alpha_2 \cos\alpha_3 & \sin\alpha_3 & 0 \\ 0 & \cos\alpha_2 & 0 & 1 \end{Vmatrix}, \quad \boldsymbol{\alpha}^{\cdot} = \begin{Vmatrix} \alpha_0^{\cdot} \\ \alpha_1^{\cdot} \\ \alpha_2^{\cdot} \\ \alpha_3^{\cdot} \end{Vmatrix}$$

Hence one can see that $\alpha_1^{\cdot}, \alpha_2^{\cdot}, \alpha_3^{\cdot}$ are simply the angular velocities of precession, nutation and rotation, respectively.

Using a Legendre transformation $\boldsymbol{\alpha}^{\cdot} \mapsto \boldsymbol{\beta}$, where

$$\boldsymbol{\beta} = \partial T / \partial \boldsymbol{\alpha}^{\cdot} = B^* A B \boldsymbol{\alpha}^{\cdot}, \quad \boldsymbol{\alpha}^{\cdot} = B^{-1} A^{-1} \left(B^{-1} \right)^* \boldsymbol{\beta}$$

the kinetic energy can be expressed in terms of the generalized momenta β. We have (cf. [9, 10])

$$T = \frac{1}{2} \left(B^{-1} A^{-1} \left(B^{-1} \right)^* \boldsymbol{\beta}, \boldsymbol{\beta} \right) = \frac{1}{4A_0} \beta_0^2 + \frac{1}{2A_1} \left(\beta_1 \frac{\sin\alpha_3}{\sin\alpha_2} + \beta_2 \cos\alpha_3 - \right.$$

$$\left. - \beta_3 \frac{\cos\alpha_2 \sin\alpha_3}{\sin\alpha_2} \right)^2 + \frac{1}{2A_2} \left(\beta_1 \frac{\cos\alpha_3}{\sin\alpha_2} - \beta_2 \sin\alpha_3 - \beta_3 \frac{\cos\alpha_2 \cos\alpha_3}{\sin\alpha_2} \right)^2 + \frac{1}{2A_3} \beta_3^2 \tag{3.1}$$

Since the force function V is independent of α_0, and so moreover depends on the direction cosines, which are one-to-one functions on $SO(3)$, it follows that this variable is cyclic by (3.1). Therefore, the equations of motion can be separated with respect to the variables α_k, β_k ($k = 1, 2, 3$) and have the form known in rigid body dynamics. As a consequence, any sphere $S^3_R \subset M^4$ will be an invariant manifold. We recall that the local map $g : S^3 \to SO(3)$ is a diffeomorphism. Unlike $SO(3)$, the motion in M^4 can be described globally in a single chart, namely, M^4 itself. This entails the absence of any defects of the charts of the configuration space $SO(3)$ described by the angles of orientation angles of the body (Euler, aircraft, etc.). As the solution approaches the boundary of such a chart it becomes necessary to change to a new chart in which to describe the motion by more regular functions.

4. THE EULER CASE

Let $(\mathbf{q}^0, \mathbf{I}^0)$ be the phase space vector at the initial instant of time $t = t_0$. Bearing in mind applications in perturbation theory, we shall find formulae ensuring that this vector can be computed in the course of the motion: $(\mathbf{q}^0, \mathbf{I}^0) \mapsto (\mathbf{q}(t), \mathbf{I}(t))$. In the Euler case $\mathbf{M}(t) \equiv \mathbf{0}$. Therefore, the dynamical equation can be obtained from (2.5) in the form [11]

$$\dot{I_0} = 0$$

$$\dot{I_1} = a_1 I_2 I_3, \quad a_1 = (A_2 - A_3)/(A_2 A_3) \quad (1\ 2\ 3) \tag{4.1}$$

The first equation immediately gives the integral $I_0(t) = I_0^0$ corresponding to the cyclic variable α_0.

We shall consider the generic case when all principal moments of inertia are pairwise distinct: $a_k \neq 0$ ($k = 1, 2, 3$). Then the last three equations in (4.1) can be represented in the following traditional form

$$\frac{I_1 dI_1}{a_1} = \frac{I_2 dI_2}{a_2} = \frac{I_3 dI_3}{a_3} = I_1 I_2 I_3 dt$$

Using the differential equation

$$d\tau/dt = I_1 I_2 I_3 \tag{4.2}$$

we introduce the new independent variable, τ, which easily enables us to obtain the quadratures

$$I_k^2 = 2a_k \tau + \left(I_k^0\right)^2, \quad k = 1,2,3 \tag{4.3}$$

To fix our ideas, we will assume that τ is equal to zero at the initial time τ_0. For $\tau(t)$ we have Eq. (4.2) in the form

$$\left(\frac{d\tau}{dt}\right)^2 = \left(2a_1\tau + \left(I_1^0\right)^2\right)\left(2a_2\tau + \left(I_2^0\right)^2\right)\left(2a_3\tau + \left(I_3^0\right)^2\right) \tag{4.4}$$

Changing to the new independent variable $u = (2a_1a_2a_3)^{1/2}t$, we obtain the equation

$$\left(\frac{d\tau}{du}\right)^2 = 4(\tau - e_1)(\tau - e_2)(\tau - e_3) \tag{4.5}$$

$$e_k = \varepsilon_k - e, \quad \varepsilon_k = -\left(I_k^0\right)^2 / 2a_k, \quad k = 1,2,3, \quad e = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)/3$$

defining a doubly periodic meromorphic (elliptic) function of the complex argument, namely, the Weierstrass $\wp$-function [12] $\tau = \wp(u)$ such that the half-periods ω and ω' can be defined by the following equations

$$\wp(\omega) = e_1, \quad \wp(\omega') = e_2, \quad \wp(\omega + \omega') = e_3$$

The solution of Eq. (4.4) can therefore be given as

$$\tau = \wp\left(\left(2a_1a_2a_3\right)^{1/2} t + u_0\right)$$

where u_0 is chosen in such a way that $\wp((2a_1a_2a_3)^{1/2}t_0 + u_0) = -e$.

In the case when one of the quantities $a_k = 0$, the corresponding component of the moment-of-momentum vector is constant: $I_k(t) = I_k^0$. Suppose, for example, that $a_3 = 0$. Then the case of dynamic symmetry $A_1 = A_2$ occurs. The second and third equations in (4.1) describe the uniform rotation of the moment-of-momentum vector about the axis Ox_3 and can be reduced to the form

$$I_1^{\cdot} = -aI_2, \quad I_2^{\cdot} = aI_1 \quad \left(a = (A_3 - A_1)/(A_3A_1)I_3^0\right)$$

The solution of the Cauchy problem for this system is known to be

$$\begin{Vmatrix} I_1 \\ I_2 \end{Vmatrix} = \begin{Vmatrix} \cos at & -\sin at \\ \sin at & \cos at \end{Vmatrix} \begin{Vmatrix} I_1^0 \\ I_2^0 \end{Vmatrix}$$

In the generic situation $a_k \neq 0$ $(k = 1, 2, 3)$ the components of the moment-of-momentum vector (the solution of the Cauchy problem for (4.1)) can be represented in terms of the theta-functions [12]

$$I_k = \left[2a_k(\wp(u) - e_k)\right]^{1/2} = \frac{\theta_1'(0)\theta_{k+1}(v)}{2\omega\theta_{k+1}(0)\theta_1(v)}, \quad k = 1,2,3; \quad \theta_4 \equiv \theta_0, \quad v = \frac{u}{2\omega}$$

In these formulae the dependence on the initial data is realized through the parameters ω and ω', which, in turn, can be uniquely expressed in terms of e_k $(k = 1, 2, 3)$ [12].

Proceeding to the discussion of the kinematic equations (2.6), in what follows we shall assume that the quasivelocities $\omega_k(t)$ $(k = 1, 2, 3)$ are unknown functions of time. In the Euler case $\omega_k(t)$ are either constant (permanent rotations) or asymptotic (separatrices) or periodic (generic situation) functions of time.

Following tradition, we shall use the angular momentum integral. In $\mathbf{H}$ it has the form

$$\mathbf{q} \circ \mathbf{I} \circ \mathbf{q}^{-1} = I_0\mathbf{1} + \mathbf{G}, \quad \mathbf{G} = G_1\mathbf{i}_1 + G_2\mathbf{i}_2 + G_3\mathbf{i}_3$$

(the quaternion $\mathbf{G}$ consists of the components of the moment-of-momentum vector projected on to the stationary axes). Using (2.5) and (2.6), it is easy to check by direct differentiation that $\mathbf{G}$ is constant. This quaternion can be computed in terms of the initial data by

$$\mathbf{G} = \mathbf{q}^0 \circ \mathbf{I}^0 \circ \left(\mathbf{q}^0\right)^{-1} - I_0^0\mathbf{1}$$

We shall change to a new stationary of system coordinates in which the unit quaternion $\mathbf{g} = g_1\mathbf{i}_1 + g_2\mathbf{i}_2 + g_3\mathbf{i}_3 = \mathbf{G}/G$ $(G = |\mathbf{G}|)$ coincides with the third coordinate unit vector. To this end it suffices to perform two rotations of the original trihedron: by the precession angle ψ and by the nutation angle θ determined from the equations

$$\cos\psi = -(\mathbf{g}, \mathbf{i}_2), \quad \sin\psi = (\mathbf{g}, \mathbf{i}_1); \quad \cos\theta = (\mathbf{g}, \mathbf{i}_3), \quad \sin\theta = \left(1 - \cos^2\theta\right)^{1/2}$$

We know [4] that the quaternion of the transition to the new stationary system of coordinates can be computed from

$$\mathbf{a} = \exp\left(\frac{\psi}{2}\mathbf{i}_3\right) \circ \exp\left(\frac{\theta}{2}\mathbf{i}_1\right)$$

where $\exp(\alpha\mathbf{j}) = \cos\alpha\mathbf{1} + \sin\alpha\mathbf{j}$ is the quaternion corresponding to the rotation by 2α about $\mathbf{j}$. Then the formulae for the transition to the half-angle should be used.

Now, to describe the motion with respect to the new stationary system of coordinates we should use the homomorphism $g : \mathbf{H} \to SO(3)$ to change in (2.6) to a new unknown function $\mathbf{z}$ such that $\mathbf{q} = \mathbf{a} \circ \mathbf{z}$. Since the quaternion $\mathbf{a}$ is constant, $\mathbf{z}(t)$ satisfies the linear equation (2.6), just as $\mathbf{q}(t)$ does. The initial conditions for the desired solution of the Cauchy problem should be stated in the form

$$\mathbf{z}^0 = \mathbf{a}^{-1} \circ \mathbf{q}^0$$

where $\mathbf{a}^{-1} = \bar{\mathbf{a}}$, since $\mathbf{a} \in S^3$, where the bar denotes the complex conjugate.

To complete the integration process it remains to complete one quadrature. To do this we represent the integral for the conservation of moment of momentum in the moving of system coordinates in the form

$$\mathbf{z}^{-1} \circ \mathbf{i}_3 \circ \mathbf{z} = \mathbf{i}(t), \quad \mathbf{i}(t) = \gamma_1 \mathbf{i}_1 + \gamma_2 \mathbf{i}_2 + \gamma_3 \mathbf{i}_3 = \left(\mathbf{I}(t) - I_0^0 \mathbf{1}\right)/G \tag{4.6}$$

(the unit quaternion $\mathbf{i}(t)$ is a unit vector defining the direction of the moment-of-momentum vector in the moving frame).

For any fixed t the solution of (4.6) is determined, apart from the centralizer $\mathrm{St}(\mathbf{i}_3)$ of $\mathbf{i}_3 \in \mathbf{H}$ in the multiplicative group of $\mathbf{H}$. All quaternions of the centralizer satisfy the equation

$$\mathbf{x}^{-1} \circ \mathbf{i}_3 \circ \mathbf{x} = \mathbf{i}_3$$

and, as can easily be checked, they have the form $\mathbf{x} = x_0\mathbf{1} + x_3\mathbf{i}_3$, which corresponds to rotation about $\mathbf{i}_3$ in the case when $|\mathbf{x}| = 1$. The general solution of (4.6) can be obtained from the formula $\mathbf{z} = \mathbf{x} \circ \mathbf{y}$, where $\mathbf{x} \in \mathrm{St}(\mathbf{i}_3)$ and $\mathbf{y}$ is any partial solution of this equation.

Indeed, if $\mathbf{z}$ is the general solution and $\mathbf{y}$ is a partial solution of (4.6), then $\mathbf{z}^{-1} \circ \mathbf{i}_3 \circ \mathbf{z} = \mathbf{y}^{-1} \circ \mathbf{i}_3 \circ \mathbf{y}$. Therefore $(\mathbf{z} \circ \mathbf{y}^{-1}) \circ \mathbf{i}_3 \circ (\mathbf{z} \circ \mathbf{y}^{-1}) = \mathbf{i}_3$, i.e. $(\mathbf{z} \circ \mathbf{y}^{-1}) \in \mathrm{St}(\mathbf{i}_3)$. Fixing some partial solution $y(t)$, we obtain a differential equation in $\mathrm{St}(\mathbf{i}_3)$, which makes it possible to obtain the desired quadrature.

Equation (4.6) is equivalent to

$$\mathbf{i}_3 \circ \mathbf{z} - \mathbf{z} \circ \mathbf{i}(t) = \mathbf{0}$$

or the linear algebraic system

$$\begin{Vmatrix} 0 & \gamma_1 & \gamma_2 & \gamma_3 - 1 \\ -\gamma_1 & 0 & -\gamma_3 - 1 & \gamma_2 \\ -\gamma_2 & \gamma_3 + 1 & 0 & -\gamma_1 \\ -\gamma_3 + 1 & -\gamma_2 & \gamma_1 & 0 \end{Vmatrix} \begin{Vmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Vmatrix}$$

which for $\gamma_3 \neq 1$ defines a two-dimensional linear space with basis vectors

$$\mathbf{y}^1 = \mathbf{i}_1 - \mathbf{i}_2 \circ \mathbf{i}(t), \quad \mathbf{y}^2 = \mathbf{i}_1 + \mathbf{i}_2 \circ \mathbf{i}(t)$$

It follows that any element of this space has the form $\mathbf{y} = \mathbf{c} + \mathbf{d} \circ \mathbf{i}(t)$, where $\mathbf{c} = y_1\mathbf{i}_1 + y_2\mathbf{i}_2$ and $\mathbf{d} = y_2\mathbf{i}_1 - y_1\mathbf{i}_2$ for any $y_1, y_2 \in \mathbf{R}$. Substituting $\mathbf{z} = \mathbf{x} \circ (\mathbf{c} + \mathbf{d} \circ \mathbf{i}(t))$, where $\mathbf{x}$ into (2.6) and using the identities

$$\mathbf{i}_1 \circ \mathbf{q} \circ \mathbf{i}_1 + \mathbf{i}_2 \circ \mathbf{q} \circ \mathbf{i}_2 = -2q_0\mathbf{1} + 2q_3\mathbf{i}_3, \quad \mathbf{i}_1 \circ \mathbf{q} \circ \mathbf{i}_2 - \mathbf{i}_2 \circ \mathbf{q} \circ \mathbf{i}_1 = 2q_3\mathbf{1} + 2q_0\mathbf{i}_3$$

we obtain the equation

$$\mathbf{x}^{\cdot} = \mathbf{x} \circ \boldsymbol{\delta}(t), \quad \left(\boldsymbol{\delta}(t) = \alpha(t)\mathbf{1} + \beta(t)\mathbf{i}_3\right) \tag{4.7}$$

where

$$\alpha(t) = \left(\omega_0 + \omega_2\gamma_1 - \omega_1\gamma_2 - \omega_0\gamma_3\right)\left[2(1-\gamma_3)\right]^{-1}$$

$$\beta(t) = \left(\omega_1\gamma_1 + \omega_2\gamma_2 + \omega_3\gamma_3 - \omega_3\right)\left[2(1-\gamma_3)\right]^{-1}$$

We have obtained a differential equation in $\mathrm{St}(\mathbf{i}_3)$, since $\mathbf{x}(t) \in \mathrm{St}(\mathbf{i}_3)$ for any t. But the centralizer $\mathrm{St}(\mathbf{i}_3)$ of $\mathbf{i}_3$ is a subalgebra in $\mathbf{H}$ with generators $\mathbf{1}$, $\mathbf{i}_3$. Therefore, on $\mathrm{St}(\mathbf{i}_3)$ there is a unique structure of the field of complex numbers, and $\mathrm{St}(\mathbf{i}_3) \simeq \mathbf{C}$.

Equation (4.7) can be interpreted in the complex sense so that $\mathbf{i}_3 = \sqrt{-1}$. Commutativity holds and the desired quadrature has the form

$$\mathbf{x}(t) = \mathbf{x}^0 \circ \exp\int_{t_0}^{t}\left(\alpha(\tau)\mathbf{1} + \beta(\tau)\mathbf{i}_3\right)d\tau, \quad \mathbf{x}^0 = \mathbf{z}^0 \circ \frac{-\mathbf{c} + \mathbf{i}(t_0) \circ \mathbf{d}}{\left|\mathbf{c} + \mathbf{d} \circ \mathbf{i}(t_0)\right|^2}$$

Note that for a real motion the antiderivative of $\alpha(t)$ in the last quadrature is a purely periodic function in the generic situation.

Indeed, for a real motion $\omega_0 = 0$. Therefore

$$\alpha(t) = \frac{1}{2}(\omega_2\gamma_1 - \omega_1\gamma_2) = \frac{A_1 - A_2}{2G}\omega_1\omega_2$$

and $\omega_1(t)\omega_2(t)$ has zero mean in the Euler case, which can easily be derived because the factors are even.

If, unlike above, the condition $\gamma_3(t') = 1$ is satisfied at some time t', then by the dynamical equations (2.5) this condition will be satisfied at any time during the motion, i.e. permanent rotation will occur. The identities $\gamma_1(t) \equiv 0$, $\gamma_2(t) \equiv 0$, $\gamma_3(t) \equiv 1$ hold in this case. Therefore, the kinematic equation (2.6) takes the form

$$\mathbf{z}^{\cdot} = \frac{1}{2}\mathbf{z} \circ \frac{I_3^0}{A_3}\mathbf{i}_3 \tag{4.8}$$

and the integral of conservation of moment of momentum is

$$\mathbf{z}^{-1} \circ \mathbf{i}_3 \circ \mathbf{z} = \mathbf{i}_3$$

This means that $\mathbf{z}(t) \in \mathrm{St}(\mathbf{i}_3)$. As above, Eq. (4.8) can be integrated in the complex sense and the solution

$$\mathbf{z}(t) = \mathbf{z}^0 \circ \exp\left(\frac{I_3^0}{2A_3}(t - t_0)\mathbf{i}_3\right)$$

can be obtained as a uniform rotation about the unit vector $\mathbf{i}_3$.

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